Exam Calculus 2

9 April 2024, 18:15-20:15

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points. Calculators, books and notes are not permitted.

1. **[6+6+8=20 Points]** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \frac{x^2y+y^3+2x^2+2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases},$$

where $c \in \mathbb{R}$.

(a) Determine c such that f becomes continuous at $(x,y) = (0,0)$.

(b) For the value of c found in part (a) and $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ a unit vector, determine the directional derivative $D_{\boldsymbol{u}}f(0,0)$.

(c) Use the definition of differentiability to show that for the value of c found in part (a), the function f is differentiable at $(x,y) = (0,0)$ and determine the derivative of f at $(x,y) = (0,0)$.

2. **[5+8+9=22 Points]** The surface $S \subset \mathbb{R}^3$ given by the equation

$$4x^2 + y^2 - 4z^2 - 8x + 2y + 4z + 8 = 0$$

is a hyperboloid of two sheets which contains the point $(x_0, y_0, z_0) = (2,0,2)$.

(a) Find the tangent plane at the point (x_0, y_0, z_0) using the fact that S is the level set of a suitable function $g : \mathbb{R}^3 \to \mathbb{R}$.

(b) Use the Implicit Function Theorem to show that near the point (x_0, y_0, z_0), the surface S can be considered to be the graph of a function f of the variables x and z. Compute the partial derivatives f_x and f_z at (x_0, z_0) and show that the tangent plane found in part (a) coincides with the graph of the linearization of f at (x_0, z_0).

(c) Use the method of Lagrange multipliers to find the point(s) in S closest to the (x,y)-plane.

— **please turn over** —

3. **[6+12+5=23 Points]** Consider the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\mathbf{F}(x, y, z) = (2x + z)\cos(x^2 + xz)\,\mathbf{i} - (z + 1)\sin(y + yz)\,\mathbf{j} + \big(x\cos(x^2 + xz) - y\sin(y + yz)\big)\,\mathbf{k}$$

for each $(x, y, z) \in \mathbb{R}^3$.

(a) Show that $\mathbf{F}$ is conservative.

(b) Determine a potential function for $\mathbf{F}$.

(c) Let $C \in \mathbb{R}^3$ be the curve with parametrization $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + (\pi t - \sin\frac{\pi t}{2})\,\mathbf{k}$ with $0 \leq t \leq 1$. Compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$ with C oriented by the tangent vectors associated with the parametrization $\mathbf{r}$.

4. **[3+22=25 Points]** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as

$$\mathbf{F}(x, y, z) = \mathbf{k} \times (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

for $(x, y, z) \in \mathbb{R}^3$. Let S be the surface $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1,\ y = z\}$ oriented by the normal vector $(0, -1, 1)$.

(a) Sketch the surface S and mark the induced orientation on the boundary ∂S of S.

(b) Verify Stokes's theorem for $\mathbf{F}$ and S by computing both sides of the equality

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

Solutions

1. (a) In order to determine the limit of f at $(x,y)=(0,0)$ we use polar coordinates $(x,y)=(r\cos\theta, r\sin\theta)$ for $(x,y)\neq(0,0)$. Then

$$\begin{aligned} f(x,y) &= \frac{r^3\cos^2\theta\sin\theta + r^3\sin^3\theta + 2r^2\cos^2\theta + 2r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= r(\cos^2\theta\sin\theta + \sin^3\theta) + 2. \end{aligned}$$

Considering the limit $r\to 0$ yields that f becomes continuous at $(x,y)=(0,0)$ for $c=2$.

(b) Let $\boldsymbol{u}=(v,w)\in\mathbb{R}^2$ with $v^2+w^2=1$. Then

$$\begin{aligned} D_{\boldsymbol{u}}f(0,0) &= \lim_{h\to 0}\frac{f(hv,hw)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{h^3v^2w+h^3w^3+2h^2v^2+2h^2w^2}{h^2(v^2+w^2)}-2\right) \\ &= \lim_{h\to 0}\frac{1}{h^3}\left(h^3v^2w+h^3w^3+2h^2-2h^2\right) \\ &= \lim_{h\to 0} \\ &= v^2w+w^3 \\ &= w(v^2+w^2) \\ &= w, \end{aligned}$$

where in the third and last equality we used $v^2+w^2=1$.

(c) According to part (b) we have $f_x(0,0)=0$ (choose $\boldsymbol{u}=(v,w)=(1,0)$) and $f_y(0,0)=1$ (choose $\boldsymbol{u}=(v,w)=(0,1)$). So the linearization of f at $(x,y)=(0,0)$ is given by

$$L(x,y)=f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=2+y.$$

For the differentiability of f at $(0,0)$ we have that the limit of

$$\frac{f(x,y)-L(x,y)}{\|(x,y)-(0,0)\|}$$

is 0 for $(x,y)\to(0,0)$. For $(x,y)\neq(0,0)$, we have

$$\begin{aligned} \frac{f(x,y)-L(x,y)}{\|(x,y)-(0,0)\|} &= \frac{1}{(x^2+y^2)^{1/2}}\left(\frac{x^2y+y^3+2x^2+2y^2}{x^2+y^2}-(2+y)\right) \\ &= \frac{1}{(x^2+y^2)^{3/2}}\left(x^2y+y^3+2x^2+2y^2-2(x^2+y^2)-y(x^2+y^2)\right) \\ &= \frac{1}{(x^2+y^2)^{3/2}}(0) \end{aligned}$$

which converges to 0 as $(x,y)\to(0,0)$. The function f is hence differentiable at $(x,y)=(0,0)$.

The derivative is

$$\nabla f(0,0)=(f_x(0,0),f_y(0,0))=(0,1).$$

2. (a) $g(x,y,z) := 4x^2 + y^2 - 4z^2 - 8x + 2y + 4z + 8$

$\Rightarrow \nabla g(x,y,z) = (8x-8,\ 2y+2,\ -8z+4)$

$\nabla g(2,0,2) = (16-8,\ 2,\ -16+4)$

$= (8,\ 2,\ -12)$

tangent plane of S' at $(x_0,y_0,z_0) = (2,0,2)$:

$\nabla g(x_0,y_0,z_0) \cdot (x-x_0,\ y-y_0,\ z-z_0) = 0$

$\Leftrightarrow (8,2,-12) \cdot (x-2,\ y,\ z-2) = 0$

$\Leftrightarrow 8x - 16 + 2y - 12z + 24 = 0$

$\Leftrightarrow 4x + y - 6z = -4$

(b) $\frac{\partial g}{\partial y}(x_0,y_0,z_0) = 2y_0 + 2 = 2 \neq 0$

$\Rightarrow$ by IFT $\exists f : U \subset \mathbb{R}^2 \to \mathbb{R}$ ~~s.t.~~ with

U neighb. of $(x_0,z_0) \in \mathbb{R}^2$ s.t.

If (x,y,z) satisfies $y = f(x,z)$ for $x,z \in U$ and $y \in V$, $\mathbb{R} \supset V$ neighb. of y_0

$\Leftrightarrow g(x,y,z) = 0$

moreover:

$f_x(x_0,z_0) = -\frac{g_x(x_0,y_0,z_0)}{g_y(x_0,y_0,z_0)} = -\frac{8}{2} = -4$

$$f_z(x_0,z_0) = -\frac{g_z(x_0,y_0,z_0)}{g_y(x_0,y_0,z_0)} = -\frac{-12}{2} = 6$$

linearization of f at (x_0,z_0)

$$L(x,z) = f(x_0,z_0) + f_x(x_0,z_0)(x-x_0) + f_z(x_0,z_0)(z-z_0)$$

$$= 0 \ \cancel{+} - 4(x-2) + 6(z-2)$$

graph of L:

$$y = -4x + 8 + 6z - 12$$

$$\iff \quad 4x + y - 6z = -4$$

which agrees with part (a).

(c) let $(x,y,z) \in \mathbb{R}^3$. The square distance of (x,y,z) to the (x,y) plane is $z^2 =: d(x,y,z)$ restricted to S.

For extremal points of (x,y,z) $\exists\ \lambda \in \mathbb{R}$ s.t.

$$\nabla g(x,y,z) = \lambda \nabla d(x,y,z)$$

with $(x,y,z) \in S$. This gives the set of equations:

$$\begin{cases} 8x - 8 = \lambda \cdot 0 \\ 2y + 2 = \lambda \cdot 0 \\ -8z + 4 = 2\lambda z \\ g(x,y,z) = 0 \end{cases} \iff \begin{cases} x = 1 \\ y = -1 \\ -8z + 4 = 2\lambda z \\ g(x,y,z) = 0 \end{cases}$$

$$4 + 1 - 4z^2 - 8 - 2 + 4z + 8 = 0$$

$$\Leftrightarrow \quad +4z^2 \mp 4z = +3$$

$$\Leftrightarrow \quad 2z - 1 = \pm 2$$

$$\Leftrightarrow \quad z = \frac{1}{2}(1 \pm 2) = \begin{cases} 3/2 \\ -1/2 \end{cases}$$

$\Rightarrow$ extrema at $(x,y,z) = \cancel{\left(1,-1,\frac{1}{2}(1 \pm \sqrt{3})\right)}$

$$\left(1, -1, \begin{matrix} 3/2 \\ -1/2 \end{matrix}\right)$$

fill in $(1,-1,\frac{3}{2})$ into $d = \left(\frac{3}{2}\right)^2 = \frac{9}{4} \longrightarrow$ max

$(1,-1,-\frac{1}{2})$ into $d = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \longrightarrow$ min

both are minima
(two sheets)

(a) Curl $\mathbf{F} = 0$

Solutions

3. (b) Writing the vector field $\mathbf{F}$ as

$$\mathbf{F} = Q(x,y,z)\,\mathbf{i} + P(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$$

the vector equation $\nabla f = \mathbf{F}$ has the components

$$\frac{\partial f}{\partial x} = Q(x,y,z) = (2x+z)\cos(x^2+xz), \tag{1}$$

$$\frac{\partial f}{\partial y} = P(x,y,z) = -(z+1)\sin(y+yz), \tag{2}$$

$$\frac{\partial f}{\partial z} = R(x,y,z) = x\cos\left(x^2+xz\right) - y\sin\left(y+yz\right). \tag{3}$$

Integrating Equation (1) with respect to x gives

$$\begin{aligned} f(x,y,z) &= \int (2x+z)\cos(x^2+xz)\,\mathrm{d}x \\ &= \int \cos u\,\mathrm{d}u \\ &= \sin(x^2+xz) + g(y,z), \end{aligned}$$

where in between we substituted $u = x^2+xz$ and $g(y,z)$ stands for an integration constant which can depend on y and z. Filling in $f(x,y,z) = \sin(x^2+xz) + g(y,z)$ in Equation (2) gives

$$\begin{aligned} g(y,z) &= \int -(z+1)\sin(y+yz)\,\mathrm{d}y \\ &= -\int \sin u\,\mathrm{d}u \\ &= \cos(y+yz) + h(z), \end{aligned}$$

where in between we substituted $u = y+yz$ and $h(z)$ is an integration constant which can depend on z. Thus f becomes

$$f(x,y,z) = \sin(x^2+xz) + \cos(y+yz) + h(z)$$

which when filled into Equation (3) gives

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[\sin(x^2+xz) + \cos(y+yz) + h(z)] \\ &= x\cos(x^2+xz) - y\sin(y+yz) + \frac{d}{dz}h(z). \end{aligned}$$

Equating the latter with $R(x,y,z)$ shows that $\frac{d}{dz}h(z) = 0$, i.e. $h(z) = c$ for some $c \in \mathbb{R}$. The potential function thus is

$$f(x,y,z) = \sin(x^2+xz) + \cos(y+yz) + c.$$

(b) As $\mathbf{F}$ is conservative with potential function f found in part (a) we have by the fundamental theorem of line integrals

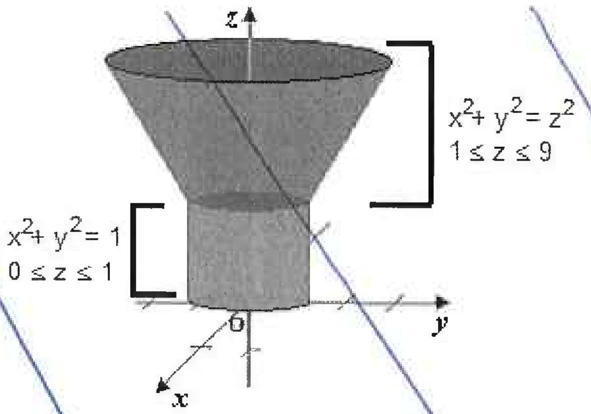
$$\int_{\mathbf{x}} \mathbf{F}(x,y,z)\cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)\,dt = f(\mathbf{x}(1)) - f(\mathbf{x}(0)).$$

$$f(\mathbf{x}(t)) = \sin(t^6 + t^3(\pi t - \sin(\frac{\pi t}{2}))) + \cos(t^2 + t^2(\pi t - \sin(\frac{\pi t}{2}))) + c.$$

Thus,

$$\begin{aligned} f(\mathbf{x}(1)) - f(\mathbf{x}(0)) &= (\sin\pi + \cos\pi) - (\sin 0 + \cos 0) \\ &= -1 - 1 = -2. \end{aligned}$$

Thus, we conclude that $\int_{\mathbf{x}} \mathbf{F}(x,y,z)\cdot d\mathbf{s} = -2$.



2. (a)

(b) Noticing that the cone shaped region is given by $f(x,y,z) = x^2+y^2-z^2 = 0$ we can find a normal vector at (x,y,z) on the cone as $\nabla f(x,y,z) = 2x\,\mathbf{i}+2y\,\mathbf{j}-2z\,\mathbf{k}$. Normalizing this vector gives the unit normal vector

$$\mathbf{n}_1 = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\,\mathbf{i} + y\,\mathbf{j} - z\,\mathbf{k})$$

which indeed is pointing outward. Alternatively the normal vector can be compute from a parametrization $\mathbf{X}$ of the cone. Using cylinder coordinates we can choose

$$\mathbf{X}(\theta, r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r\,\mathbf{k}$$

with $0 \le \theta \le 2\pi$ and $1 \le r \le 9$. For the normal vector we then find

$$\frac{\partial \mathbf{X}}{\partial \theta} \times \frac{\partial \mathbf{X}}{\partial r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} - r\mathbf{k}$$

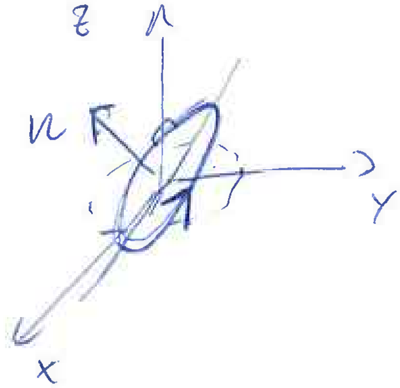
which has norm $\sqrt{2}r$. Normalizing this vector agrees with $\mathbf{n}_1$ found above.

4.

$$F(x,y,z) = k \times (xi + yj + zk)$$

$$= x\, k \times i + y\, k \times j$$

$$= xj - yi$$

a) 

$$n = (0,-1,1)$$

b) LHS: $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = 2k$

parametrization of S':

$r: (s,t) \mapsto (s,t,t)$ $\qquad s^2+t^2 \le 1$

$\frac{\partial r}{\partial s} = (1,0,0)$ $\qquad \frac{\partial r}{\partial t} = (0,1,1)$

$$N = \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0 \cdot i - 1 \cdot j + k$$

$= k - j$ (consistent)

$$\iint_{S'} \nabla \times F \cdot dS = \iint_{s^2+t^2 \le 1} 2k \cdot (k-j)\, ds\, dt$$

$$= +2 \iint_{s^2+t^2 \le 1} ds\, dt = 2\pi$$

RHS: parametrization of ∂S:

$r(t) = (\cos t, \sin t, \sin t)$ $\quad t \in [0, 2\pi]$

r' gives the desired orientation

$r'(t) = (-\sin t, \cos t, \cos t)$

$$\oint_{\partial S} F \cdot ds = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} \cancel{-\sin t\, j +} (\cos t\, j - \sin t\, i) \cdot (-\sin t\, i + \cos t\, j + \cos t\, k)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2)\, dt$$

$$= 2\pi$$